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DETERMINATION OF OPTIMUM, TRANSIENT EXPERIMENTS FOR THERMAL CONTACT CONDUCTANCE

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Abstract—Some recent work indicates that the thermal contact conductance can vary significantly with time. In previous studies of transient cases, the analyses were restricted to cases utilizing either a thin specimen or thermocouples located at the interface. Because the contact conductance is a volume effect, these restrictions are frequently too severe. For this reason a method is presented for analyzing temperature data obtained from thermally thick specimens with the thermocouples not necessarily located at the interface.

There are a number of possible transient experiments that might be employed, but they are not all equally as effective for determining the conductance. A criterion is given to permit comparisons between experiments; a number of possible experiments are compared utilizing this criterion. From this comparison some optimum experiments are indicated. These optimum experiments permit the contact conductance to be determined more accurately than utilizing other similar experiments with the same accuracy of the temperature measurements.

NOMENCLATURE

A, dimensionless parameter, $= 4B\tau$;
m, dimensionless parameter, $= h_0L/k$;
B_x, dimensionless parameter, $= hx/k$;
c_p, specific heat at constant pressure [Btu/lbm-F];
F, sum of squares function [see equation (4)];
h, thermal contact conductance [Btu/hft^2-F];
h^i, thermal contact conductance at time t_i[Btu/hft^2 F];
i, time index corresponding to t_i;
j, space index corresponding to x_j;
k, thermal conductivity [Btu/hftF];
L, specimen thickness [ft];
L_1, thickness of left specimen [ft];
L_2, thickness of right specimen [ft];
t, time [h];
t_{max}, duration of the time interval for evaluating $\bar{\Delta}$ [h];
T, temperature [°F];
$\bar{T}$, dimensionless temperature [see equation (12a)];
T_{max}, maximum temperature in the experiment;
T_{min}, minimum temperature in the experiment;
x, coordinate [ft].

Greek symbols
α, thermal diffusivity $= k/\rho c_p$[ft^2/h];
$\bar{\Delta}$, optimum criterion given by equation (10);
ϵ, small number such as 0·01;
Δh_e, error in h [see equation (9)];
$\bar{\Delta}_{max}$, maximum value of $\bar{\Delta}$ for a given experiment;
ρ, density [lbm/ft^3];
τ, dimensionless time, $= \alpha t/L^2$;
τ_m, dimensionless time for maximum $\bar{\Delta}$;

τ_x, dimensionless time for a semi-infinite body, $= \alpha t/x^2$;

$\tau_{x,m}$, dimensionless time for maximum $\bar{\Delta}$ for semi-infinite body.

1. INTRODUCTION

THERE are ever increasing demands upon heat transfer engineers for more precise temperature calculations to enable devices of lower cost and weight to be built. This, in turn, requires that the thermal contact conductance between materials be understood more thoroughly and that values for the conductance be measured more accurately. Many researchers have investigated the conductance using steady-state techniques [1, 2]. Little work, however, has been done under transient conditions which is the subject of this paper. Examples for which the knowledge of the variation of the conductance with time might be important include the thermal design of re-entry vehicle heat shields, rocket nozzles, nuclear reactors, electronic equipment, gun barrels and brake drums.

A paper by Jacobs and Starr [3] reports at cryogenic temperatures for gold and copper that "a progressive decrease of conductance with time after cooling was observed." A more recent paper by Schauer and Giedt [4] also reports the contact conductance can vary significantly with time. In both cases the specimens were quite thin. Because contact conductance is not simply a surface effect but is a volume effect, their results have not been generally accepted in part because of the use of thin specimens. Analyses that we have performed using temperature data for "thick" specimens also indicate that there are transient effects.

The methods of analysis given in [3, 4] for determining the contact conductance are restricted to cases for which: (a) thin specimens are utilized or thermocouples are located at the interface, and (b) the materials have temperature-independent thermal properties. The first objective of this paper is to present an analytical method suitable for determining the contact conductance as a function of time from transient temperature measurements located inside either a thermally thick or thin body whose thermal properties can be temperature-dependent. The second objective is an analytical investigation of various optimum experiments useful for measuring the conductance.

In [5] these objectives are also covered. The first objective is treated more thoroughly herein, however. The results derived in [5] related to the second objective are utilized in the present work. In the investigation of optimum experiments the emphasis is upon cases for which the contact conductance is time invariant; it is shown, however, that this case is helpful for providing insight into the transient conductance case. There is one basic steady state experiment for determining the conductance; for the transient case the number of cases that can be suggested is large because one can independently vary the boundary conditions at either extremity and the initial conditions. Not all of these cases would be equally as efficacious for determining the conductance and hence, the need of finding optimum experiments. The inconclusive results [6, 7] regarding transient effects also suggest to us that optimum experiments may be necessary to investigate the transient conductance.

The basic procedure used herein for determining the conductance has been used to find thermal conductivity, specific heat and thermal diffusivity [8–11]; these are parameters appearing in a partial differential equation. Basically the same procedure is applied in this paper for the calculation of the contact conductance even though the latter is found in the interface condition. Parameters appearing in boundary or interface conditions—and not in the differential equation—can be functions of time while properties such as thermal conductivity are not. As a quantity appearing in a boundary condition, some of the methods given herein can be also used for calculating heat and mass transfer coefficients from transient data. The basic method which can be utilized for determining both properties and parameters appearing in boundary conditions is called nonlinear estimation [12–14]. Similar methods have been employed in such diverse fields as astronomy, water pollution, physics and chemical engineering.

Nonlinear estimation can be utilized for transient as well as steady-state situations. It has a number of advantages over the conventional method of steady-state analysis. It does not require as many thermocouples; it does not utilize extrapolation to determine the interface temperatures; and there is no inconsistency between the heat flux implied leaving one specimen with flux entering the other. Moreover, the method extends simply to treatment of time-varying conductance and can be utilized to investigate optimum experiments for determining the conductance.

2. HEAT TRANSFER PROBLEM

A typical geometry for a plane, one-dimensional case of two bodies which have a contact conductance, h, at the interface is shown in Fig. 1. The heat-conduction equations for bodies 1 and 2 are respectively

$$\frac{\partial}{\partial x}\left(k_1 \frac{\partial T_1}{\partial x}\right) = \rho_1 c_{p,1} \frac{\partial T_1}{\partial t}; \quad -L_1 < x < 0 \quad (1)$$

$$\frac{\partial}{\partial x}\left(k_2 \frac{\partial T_2}{\partial x}\right) = \rho_2 c_{p,2} \frac{\partial T_2}{\partial t}; \quad 0 < x < L_2 \quad (2)$$

where T_1 and T_2 are temperatures for materials 1 and 2. The properties are assumed known. The interface conditions are

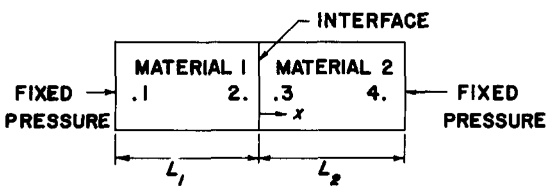

FIG. 1. Illustration of typical experiment.

$$-k_1 \frac{\partial T_1(x,t)}{\partial x}\bigg|_{x=0} = h[T_1(0,t) - T_2(0,t)] = -k_2 \frac{\partial T_2(x,t)}{\partial x}\bigg|_{x=0} \quad (3)$$

where $T_1(0, t)$ is the temperature in body 1 at the interface, etc.

The boundary conditions at $x = -L_1$ and at $x = L_2$ can be given temperatures, given heat fluxes or some other known conditions. Thermocouples can be located at positions 1, 2, 3, 4 and others (see Fig. 1).

In placing thermocouples 2 and 3 "near" the interface (Fig. 1) and using equations (1) and (2), one is assuming that the heat flow near these thermocouples is one-dimensional. If there is large-scale waviness of the mating surfaces or a lack of flatness at certain regions, etc., the heat flow might well be three-dimensional a substantial distance from the interface. It is assumed, however, that the thermocouples are placed outside this "disturbance layer".

3. NONLINEAR ESTIMATION PROCEDURE

The calculated temperatures at (x_j, t_i) are designated T_j^i are found (usually) from a finite difference solution of (1–3) with appropriate boundary conditions. By varying h, T_j^i is made to agree in a least squares sense with the measured temperatures Y_j^i. That is, the sum of squares function F for n thermocouples and measurements at m discrete times,

$$F(h) = \sum_{i=1}^{m} \sum_{j=1}^{n} [T_j^i(h) - Y_j^i]^2, \quad (4)$$

is minimized with respect to h.

The sum of squares function F can be efficiently minimized in many ways [16, 17]. A simple procedure approximates at each iteration step the calculated temperature by the Taylor series

$$T_j^i(h) \approx T_j^i(h_l) + T_{h,j}^i \,\Delta h_l, \quad (5)$$

where

$$\Delta h_l = h_{l+1} - h_l \quad (6a)$$

$$T_{h,j}^i = \left.\frac{\partial T_j^i}{\partial h}\right|_{h_l} \approx \frac{T_j^i h_l(1+\epsilon) - T_j^i(h_l)}{\epsilon h_l}. \quad (6b)$$

The derivative $T_h = \partial T/\partial h$ is called a sensitivity coefficient. (Other definitions could be used.) The T's on the right-hand side of equation (6b) are calculated with a finite-difference program. The iterative procedure begins with an estimated value for h_0, corresponding to $l = 0$.

Using $\partial F/\partial h = 0$ at the minimum value of F gives for Δh_l, after using equation (5),

$$\Delta h_l = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} T_{h,j}^i [Y_j^i - T_j^i(h_l)]}{\sum_{i=1}^{m}\sum_{j=1}^{n} (T_{h,j}^i)^2}. \quad (7)$$

This procedure can be modified readily to treat time-variable h. Instead of calculating a single h for the entire experiment, the duration of the experiment is divided into a number of time regions, for each of which a constant h is calculated. In equation (7) the summation of j, which is for the thermocouples, ranges over all the n thermocouples. The time index, i, can extend only to $m = 2$ or 3, if many h's are to be found in an effort to approximate time-varying h. Note that the finite-difference calculations for the temperature, T_j^i, might use much finer time-steps than the time intervals between successive times at which experimental temperatures are used. These finer Δt's would be used to insure accurately calculated temperatures [9].

Many other numerical approximations are possible in addition to the one mentioned in connection with (7). One could approximate h by a polynomial of first, second or other degrees. In so doing, several variations on each degree would be possible. One simple linear approximation which would use (7) is the following. For the first time region h could be considered constant. For subsequent time regions would be considered linear in time as

$$h = h^I + \frac{t - t_I}{t_{I+1} - t_I}(h^{I+1} - h^I) \quad (8)$$

where h_I is known and h_{I+1} is to be found using nonlinear estimation. The time t would be evaluated for the $I + 1$ time interval,

$$t_{I+1} - \tfrac{1}{2}(\Delta t)_{I+1} < t < t_{I+1} + \tfrac{1}{2}(\Delta t)_{I+1}.$$

The duration of the Ith time region for finding h^I is designated $(\Delta t)_I$. For the linear approximation described above, the time regions $(\Delta t)_I$ can be larger than those for h assumed constant; this is because a time-variable h can be better approximated by linear segments than constant ones.

4. ERROR ANALYSIS

An analysis of the effect of small errors in T_e helps to provide some insight into the efficacy of this method. The analysis follows rather closely some of the development given above. The result is an error in h, designated Δh_e, given by

$$\frac{\Delta h_e}{\hat{h}} = \frac{\Sigma\Sigma\, hT_{h,j}^i\, \Delta T_j^i}{\Sigma\Sigma\,(hT_{h,j}^i)^2} \quad (9)$$

where ΔT is an error in the measured temperature. T_h is evaluated for the value of h minimizing F, designated $\hat{h}$. ($\hat{h}$ is not identical to the true or correct value of h due to small errors in Y and the calculational procedure).

The errors in temperature tend to be biased rather than random. If they were random, then one could use standard statistical procedures to find the confidence region [18].

5. CRITERIA FOR OPTIMUM EXPERIMENTS

Rather than repeating an analysis to determine a criterion for optimizing experiments, which is given in [10] and [13], some of the basic conditions are stated and a correlation between the criterion and some errors in h is demonstrated.

The conditions are:

1. The errors in the temperature measurements are small.
2. The (a) number of thermocouples, (b) maximum temperature difference between the highest T of the high-T specimen and

the lowest T of the low-T specimen and (c) number of equally-spaced temperature measurements of the experiment are each fixed in the possible experiments considered below.

3. The difference between the minimum value of the sum of squares function F and another slightly larger F is fixed.

These conditions could be utilized to determine a classical confidence region for a given experiment if the errors in T were random. Because the temperature errors in our research have tended to be biased rather than random, it may not be correct to specify a classical confidence region. However, one can still derive a criterion which would indicate the relative efficacy of different possible experiments [10, 13]. The criterion for an optimum experiment—one which produces minimum errors in h for a given error distribution in T—is to maximize

$$\bar{\Delta} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{hT_{h,j}^{i}}{T_{\max} - T_{\min}} \right]^2 \quad (10a)$$

where $T_{\max}$ and $T_{\min}$ are respectively the maximum and minimum temperatures in the specimens. Note that (10a) is a normalized form of the denominator of (9). If the measurements are equally spaced in time, if m is large and if $n = 1$ (one thermocouple), then $\bar{\Delta}$ can be approximated by

$$\bar{\Delta} = \frac{1}{t_{\max}} \int_{0}^{t_{\max}} \left[\frac{hT_h(\theta)}{T_{\max} - T_{\min}} \right]^2 \mathrm{d}\theta \quad (10b)$$

where $t_{\max}$ is the duration of the time interval for evaluating $\bar{\Delta}$. The examples to be considered have two identical specimens and similar boundary conditions at $x = -L_1$ and $x = L_2$. Because of this symmetry about the interface (see equation (16) of [5]), the sensitivity coefficients in both specimens have the same absolute values at the same distance from the interface and at the same time. Hence it is convenient to write $\bar{\Delta}$ for a single thermocouple which could be considered for the following examples to be in either the high or low temperature specimen. Exactly the same value of $\bar{\Delta}$ would also be obtained using two thermocouples both the same distance from the interface with one on either side of the interface.

In [5] the sensitivity coefficient T_h is investigated and insight is provided into the types of experiments which would maximize $\bar{\Delta}$. One condition for transient experiments that maximize $\bar{\Delta}$ is to initially have unequal but uniform temperatures in the two specimens. Hence this particular condition is considered below for several boundary conditions.

This criterion specifically applies to determining a single conductance for the entire experiment. This criterion can be helpful, however, for determining optimum experiments for finding $h(t)$. Suppose h^i is the conductance between times t_i and t_{i+1}, then it can be proved that (see Appendix)

$$h_0 \frac{\partial T(x,t)}{\partial h_0} = \sum_{i=1}^{M} h^i \frac{\partial T(x,t)}{\partial h^i} \quad (11)$$

for the special case of $h^i = h_0 (i = 1, 2, \ldots, M)$ where h_0 is a constant h for the entire experiment. Now the accuracy of the determination of any h^i is affected by the magnitude of its sensitivity coefficient. Each sensitivity coefficient in (11) has the same sign (minus in the high temperature specimen and plus in the other specimen). If an experiment has been chosen to maximize $|\partial T/\partial h_0|$, then the sensitivity coefficients would be expected to be larger on the average than for another similar experiment for which $|\partial T/\partial h_0|$ had not been maximized. Hence an investigation of the optimum experiments for determinating a constant conductance would simultaneously yield information about optimum experiments for $h(t)$. Because of this conclusion and for brevity, only the constant-h case is examined below.

6. POSSIBLE OPTIMUM EXPERIMENTS

6.1. *Negligible internal resistance case* (case I)

Perhaps the simplest transient case to investigate is for two identical specimens of thickness L initially at temperatures T_{max} and T_{min}, $B \equiv hL/k \ll 1$. The temperature in the low temperature specimen is given by

$$T \equiv \frac{T - T_{min}}{T_{max} - T_{min}} = \tfrac{1}{2}(1 - e^{-2B\tau}) \quad (12a)$$

where

$$\tau = \frac{\alpha t}{L^2} \quad \text{and} \quad \alpha = \frac{k}{\rho c_p}. \quad (12b)$$

For convenience let

$$B\tau = \frac{A}{4}. \quad (12c)$$

Then differentiating (12a) with respect to B gives

$$\overline{T}_B \equiv B\frac{\partial \overline{T}}{\partial B} = B\tau\, e^{-2B\tau} = \frac{A}{4} e^{-A/2} \quad (13)$$

and thus

$$\overline{\Delta} \equiv \frac{1}{\tau_m}\int_0^{\tau_m}\left(B\frac{\partial \overline{T}}{\partial B}\right)^2 d\tau$$
$$= \frac{1}{16A}[2 - (A^2 + 2A + 2)e^{-A}]. \quad (14)$$

(The definition of $\overline{\Delta}$ used in equation (14) is the same as used in equation (10) which is used in all the following cases). These results are plotted in Fig. 2.

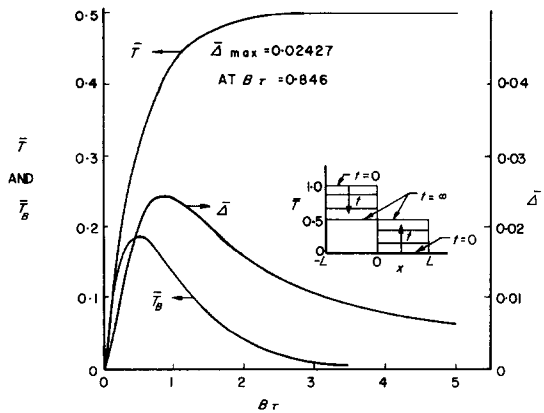

FIG. 2. Curves of $\overline{T}$, $\overline{T}_B$ and $\overline{\Delta}$ for Case I. (Finite plate with $B = hL/k \approx 0$ and constant h).

The dimensionless temperature starts at zero and reaches its maximum approximately at $B\tau = 3$ while the maximum of $\overline{T}_B$ occurs at exactly $B\tau = 0.5$. Hence, if only one instantaneous temperature reading is to be taken, it should be at this latter time. Incidentally, if one desired to perform such an experiment and desired to find a preliminary value of h, it could be readily done by setting

$$A\tau = \frac{ht}{\rho c_p L} = 0.5 \quad (15)$$

when $\overline{T} = 0.316$.

If a large number of temperature measurements at uniform time intervals were to be made, then one would maximize $\overline{\Delta}$ in order to minimize the effect of temperature errors upon the calculation of h. This would apply for this case when using one or more thermocouples. The maximum occurs at $B\,\tau \approx 0.846$. Before analyzing the data for an experiment one does not know h and hence cannot compute precisely the optimum duration for using the temperature data. However, the $\overline{\Delta}$ maximum is not sharp; if the maximum time is chosen to be between the times associated with say, $\overline{T} = 0.35$ and 0·45, little loss of accuracy will result.

6.2 *Semi-infinite case* (Cases II and III)

The other extreme value of B compared to the previous case is $B = hL/k$ equal to infinity which occurs when $L \to \infty$. The temperature distribution in the low-temperature specimen is

$$\overline{T} \equiv \frac{T - T_{min}}{T_{max} - T_{min}} = \frac{1}{2}\left\{\operatorname{erfc}\left(\frac{\tau_x^{-\frac{1}{2}}}{2}\right) - [\exp(2B_x + 4B_x^2\tau_x)]\operatorname{erfc}\left(\frac{\tau_x^{-\frac{1}{2}}}{2} + 2B_x\tau_x^{\frac{1}{2}}\right)\right\} \quad (16)$$

where

$$\tau_x = \frac{\alpha t}{x^2} = \frac{kt}{\rho c x^2}, \qquad B_x = \frac{hx}{k} \quad (17)$$

and which for the interface ($x = 0$) reduces to

$$\frac{T(0,t) - T_{\min}}{T_{\max} - T_{\min}} = \tfrac{1}{2}\{1 - [\exp(4B_x^2\tau_x)] \operatorname{erfc}(2B_x\tau_x^{\frac{1}{2}})\}. \tag{18}$$

Note that

$$B_x^2\tau_x = \frac{h^2\alpha t}{k^2} \tag{19}$$

given in equation (18) is independent of x. For any interior point, however, T is a function of both B_x and τ_x.

Depicted in Fig. 3 are $\bar{T}$, $\bar{T}_{B_x}$ and $\bar{\Delta}$ for $x = 0$ (Case II). The optimum duration of an experiment as indicated by $\bar{\Delta}$ is

$$B_x^2\tau_{x,m} = \frac{h^2\alpha t_{\text{opt}}}{k^2} \approx 0{\cdot}40. \tag{20}$$

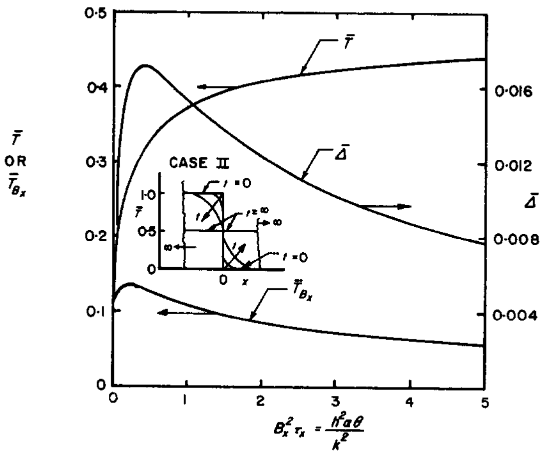

FIG. 3. Curves of $\bar{T}$, $\bar{T}_B$ and $\bar{\Delta}$ for interface for semi-infinite body (Case II).

Typical curves are depicted in Fig. 4 for the interior location indicated by $B_x = 1{\cdot}0$ (Case III). Note that the maximum value of $\bar{\Delta}$ for $x = 0$ shown in Fig. 3 is about 0·0172 while for $B_x = 1{\cdot}0$ from Fig. 4, $\bar{\Delta}_{\max}$ is the smaller value of 0·003. Figure 5 shows how $\bar{\Delta}_{\max}$ varies with B_x. The times $\tau_{x,m} = \alpha t_m/x^2$ associated with these $\bar{\Delta}_{\max}$'s are also given.

To further demonstrate the efficacy of the $\bar{\Delta}_{\max}$ criterion, the effect of two distributions of

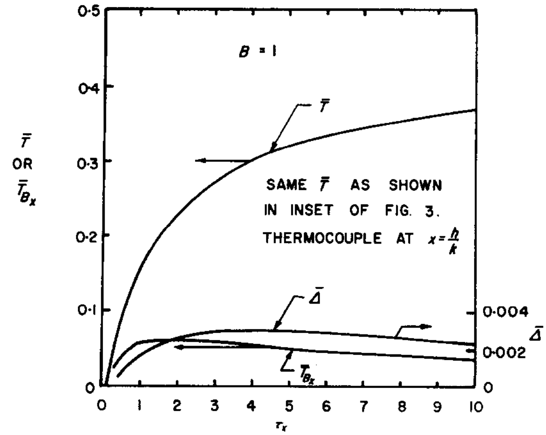

FIG. 4. Curves of $\bar{T}$, $\bar{T}_{B_x}$ and $\bar{\Delta}$ for $B = 1{\cdot}0$ for semi-infinite body (Case III).

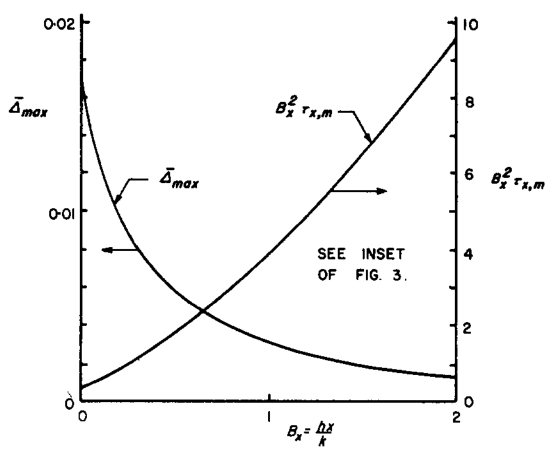

FIG. 5. Curves of $\bar{\Delta}_{\max}$ and $B_x^2\tau_{x,m}$ for semi-infinite body.

temperature measurement errors upon the accuracy of h is given. For simplicity in analysis one thermocouple is used. The first error distribution (designated "a" in Fig. 6) is for a constant error in temperature equal to

$$\Delta T^i = \epsilon(T_{\max} - T_{\min})/2 \tag{21}$$

where ϵ is a small number such as 0·01. The other curve in Fig. 10 (Curve b) is for an error which is a given fraction of the temperature rise or

$$\Delta T^i = \epsilon\bar{T}(T_{\max} - T_{\min}). \tag{22}$$

As anticipated, the per cent errors in h increase with increasing B_x which is associated with decreasing $\bar{\Delta}$. The minimum error shown is for Curve a with $B_x = 0$; if $\epsilon = 0{\cdot}01$, then the error in h would be about 2 per cent. Since some other errors indicated by Fig. 6 are much larger, it behoves one to be careful in the design of the experiments.

6.3 *Finite cases* (Cases IV, V and VI)

Three finite cases with the specimens of equal length and the same thermal properties are described in Table 1. Case IV, which has a uniform initial temperature distribution, is the easiest experiment to perform; unfortunately in most cases it is not as effective for the accurate determination of h as the other two. Case V and VI have initial temperature distributions which are uniform in each specimen at T_{max} and T_{min}.

Depicted in Fig. 7 are the $\bar{\Delta}$ versus time curves for these three cases for the special conditions of $B(= hL/k) = 0{\cdot}5$ and a thermocouple at $x = 0-$ or $0+$ (the interface). Evidently, for Case IV with $B = 0{\cdot}5$ and $x = 0$, the steady-state experiment ($\tau \to \infty$) is superior to a transient experiment of short duration (assuming h is invariant with time). For other values of B this is not true as discussed below.

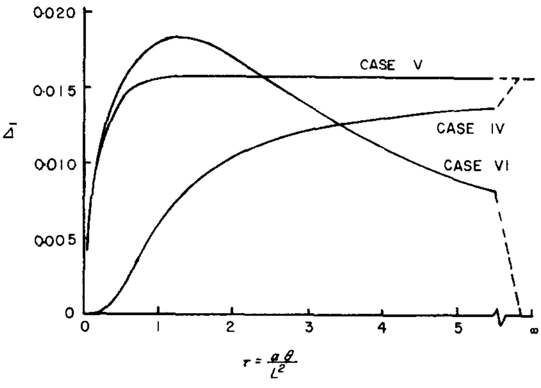

FIG. 7. $\bar{\Delta}$ vs. τ for $x = 0$ for finite body Cases IV, V and VI with $B = 0{\cdot}5$.

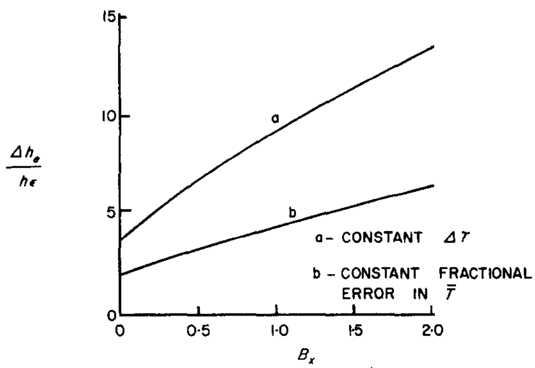

FIG. 6. Errors in h for single thermocouple in semi-infinite body.

For Cases IV the maximum values of $\bar{\Delta}$ vs. x (position of a thermocouple) are shown in Fig. 8 for $B = 0{\cdot}5, 1{\cdot}0$ and $2{\cdot}0$. These $\bar{\Delta}_{max}$-values, which are identical to those obtained in steady state, are given by

$$\bar{\Delta}_{max} = \left[1 - \frac{x}{L}\right]^2 \frac{B^2}{(2B+1)^4}. \tag{23}$$

This expression is maximized when $B = 0{\cdot}5$ for any given x/L. It can be proved that these $\bar{\Delta}_{max}$-values also apply if $T(-L, t)$ were a function of time such that $T_{min} \leqslant T(-L, t) \leqslant$

Table 1. Initial and boundary conditions for cases IV, V and VI

Case No.	Initial temperature distribution		Boundary conditions	
	$-L < x < 0$	$0 < x < L$	$x = -L$	$x = L$
IV	$T = T_{min}$	$T = T_{min}$	$T = T_{max}$	$T = T_{min}$
V	$T = T_{max}$	$T = T_{min}$	$T = T_{max}$	$T = T_{min}$
VI	$T = T_{max}$	$T = T_{min}$	$\partial T/\partial x = 0$	$\partial T/\partial x = 0$

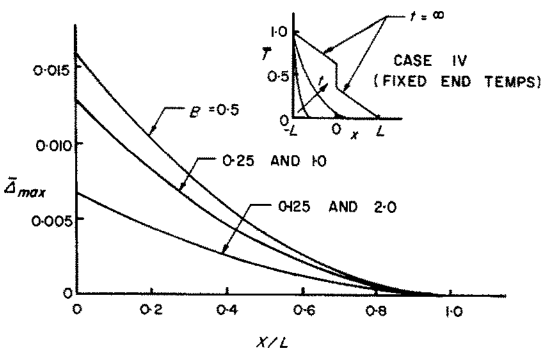

FIG. 8. $\bar{\Delta}_{max}$ for finite body, Case IV.

T_{max} and that it finally reached a constant value equal to T_{max}.

In Fig. 9 some results for Case V are shown for $\bar{\Delta}_{max}$. For a short dimensionless time the temperature response at the interface is identical to that for the two semi-infinite bodies discussed above; hence, for $B \geqslant 1$ the maximum $\bar{\Delta}$'s at the fnterface are equal to the $\bar{\Delta}_{max}$ of Fig. 3. For $B \leqslant 0{\cdot}5$ the $\bar{\Delta}_{max}$ values are given by equation (23). Because one cannot place a thermocouple at $x = 0$ without disturbing the temperature at the most critical location but rather some distance from $x = 0$, the $B = 0{\cdot}5$ value is again probably an excellent choice. For cases IV and V it is clear that one should place the thermocouples as near the interface as possible without disturbing the interface conditions.

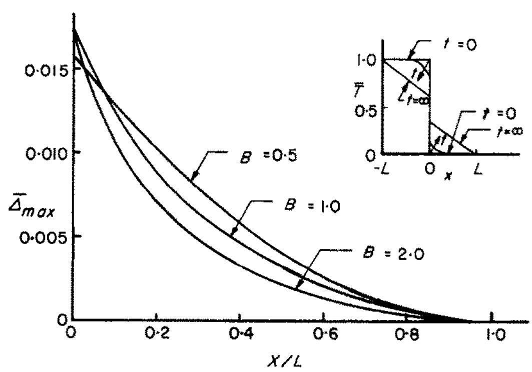

FIG. 9. $\bar{\Delta}_{max}$ for finite body, Case V.

The most interesting of the three cases is Case VI (see Fig. 10). As B approaches zero, the temperature distribution (except in the immediate vicinity of the interface) becomes uniform in x at any given time; then, $\bar{\Delta}_{max}$ becomes constant with x as $B \to 0$ which is the case discussed in Section 6.1. For the cases of $B < 0{\cdot}5$ the optimum experimental time is about $B\tau = 0{\cdot}8$ as indicated by Figs. 2 and 7. Values of $B \geqslant 1$ give values of $\bar{\Delta}_{max}$ at $x = 0$ which are identical to those for a semi-infinite body and are also shown by Fig. 9 for Case V.

For these three finite cases the optimum value of B for bodies with $T = 0$ at $x = L$ is about 0·5 as indicated by Figs. 8 and 9 while the optimum B for Case VI goes to zero. For this latter case it is not nearly as important to position the thermocouple near the interface as for the other two cases.

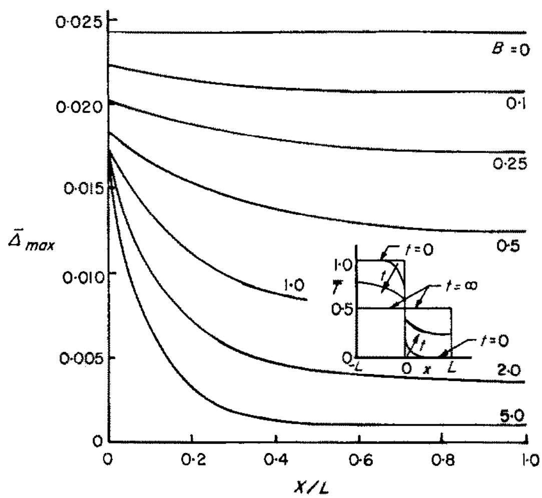

FIG. 10. $\bar{\Delta}_{max}$ for finite body, Case VI.

7. RESULTS

Since a number of cases are considered above, a summary of the results is given in Table 2. For each case the specimens on either side of the interface are identical and have the same type boundary conditions at corresponding surfaces. The $\bar{\Delta}$'s are given for only a single thermocouple. For Case III the thermocouple is

Table 2. Various optimum experiments with identical specimens on either side of interface

Case no.	Geometry	$\bar{\Delta}_{max}$	Optimum time
I	Finite bodies with $q = 0$ at $x = \pm L$ and $B \to 0$, i.e. $k \to \infty$ or $L \to 0$	0·02427	$B\tau_m = 0{\cdot}846$
II	Semi-infinite bodies for thermocouple at $x = 0+$ (or $0-$)	0·0172	$B_x^2\tau_{x,m} = 0{\cdot}40$
III	Semi-infinite bodies for interior thermocouple with $B_x = 1$	0·003	$\tau_{x,m} = 1{\cdot}7$
IV	Finite body (see Table 1) Thermocouple $x = 0+$ (or $0-$) and $B = 0{\cdot}5$	0·0156	$\tau_m \to \infty$
V	Finite body (see Table 1) Thermocouple at $x = 0+$ (or $0-$) $B \geqslant 1{\cdot}0$	0·0172	$B^2\tau_m \geqslant 0{\cdot}40$
VI	Finite body (see Table 1) Thermocouple at $x = 0+$ (or $0-$)		
VIa	$B \to 0$, i.e. $k \to \infty$ or $L \to 0$ (same as Case I)	0·02427	$B\tau_m = 0{\cdot}846$
VIb	$B = 0{\cdot}5$	0·0183	$\tau_m = 1{\cdot}25$

located inside the body while for Cases II, IV, V and VIb the thermocouple is at $x = 0-$ or $0+$.

As discussed above the optimum experiment is one in which $\bar{\Delta}$ is maximized. On this basis the optimum experiment is for two finite specimens with B tending to zero (Cases I or VIa). Figure 11 depicts the fractional errors in h due to the two error distributions discussed in Section 6.2 and used for the semi-infinite example (Fig. 6). Curve a in Fig. 11 (which is for a constant error in the temperature) has a minimum which coincides approximately with the dimensionless time at which $\bar{\Delta}$ is maximized. To some extent this is a coincidence because the minimum error could occur at smaller or greater values of B depending upon the distribution with time of the errors in the measured temperatures. However, the error distribution associated with Curve a is an important one because it is one of the most severe types of biased errors.

Also shown in Fig. 11 is Curve b which is for an error distribution which is proportional to the temperature rise. The minimum error occurs at $B\tau = 0$ which corresponds to a temperature rise (Fig. 2) of zero; this result is thus not too helpful because the errors are unlikely

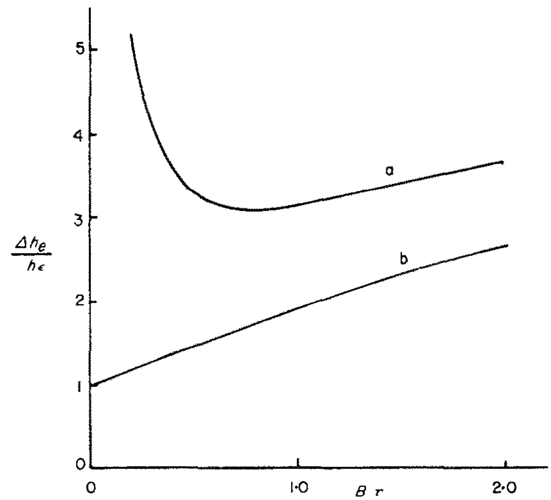

FIG. 11. Errors in h for single thermocouple in finite body, Case VI with $B \approx 0$.

to be proportional to the temperature rise as the temperature rise approaches zero.

Notice that the h-errors in Fig. 11 are less than those in Fig. 6 for $B_x = 0$. This is consistent with the $\overline{\Delta}$ criterion because the $\overline{\Delta}_{max}$ values are respectively 0·024 and 0·017. For $B_x > 0$ in Fig. 6 the h-errors increase consistently with the criterion. These results of mathematical experimentation with T-errors give further validity to the $\overline{\Delta}_{max}$ criterion for an optimum experiment.

8. SUMMARY AND CONCLUSIONS

A method utilizing nonlinear estimation is given to calculate the contact conductance h as a constant or as a function of time.

A general error analysis is given to permit the investigation of biased errors upon the calculated h; this error analysis is employed for several examples.

To aid in the determination of optimum experiments for finding a time-invariant h, the criterion $\overline{\Delta}$ is given and values for it have been determined for a number of cases.

A number of possible experiments are examined. If the two specimens are identical and a constant h is to be found, the optimum experiment is one in which the specimens are initially at different temperatures and then suddenly brought into thermal contact. The specimens are insulated at all surfaces except the interface. The dimensionless number $B = hL/k$ should be as small as conditions permit. If $B \leqslant 0{\cdot}5$, the locations of the thermocouples are not critical; if $B > 0{\cdot}5$, then some thermocouples are located as near the interface as possible but yet still outside the disturbance layer.

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APPENDIX

Derivation of Sensitivity Relation Equation (11)

Consider the geometry shown by Fig. 1; the mathematical description of the problem is given in part by equations (1–3). The initial temperature distribution is

$$T(x, 0) = T_i(x) \tag{A-1}$$

and the boundary conditions at $x = -L$ and L are

$$-k_1 \frac{\partial T_1}{\partial x}\bigg|_{x=-L} = A_1(t)\, T(-L, t) + B_1(t) \tag{A-2}$$

$$-k_2 \frac{\partial T_2}{\partial x}\bigg|_{x=L} = A_2(t)\, T(L, t) + B_2(t) \tag{A-3}$$

where A_1, A_2, B_1 and B_2 are known functions of time; any or all of these coefficients can be zero. The thermal properties are to be considered constants.

For conciseness only body 1 will be considered below although similar relations can be given for body 2.

Consider first the case of time-independent $h = h_0$ and let

$$\phi_{10} \equiv h_0 \frac{\partial T_1}{\partial h_0};\ \phi_{20} \equiv h_0 \frac{\partial T_2}{\partial h_0}. \tag{A-4}$$

Take the partial derivative of equations (1), (3) (A-1) and (A-2) with respect to h_0 and multiply by h_0 to obtain

$$k_1 \frac{\partial^2 \phi_{10}}{\partial x^2} = \rho_1 c_{p,1} \frac{\partial \phi_{10}}{\partial t} \tag{A-5}$$

$$-k_1 \frac{\partial \phi_{10}}{\partial x}\bigg|_{x=0} = h_0[\phi_{10}(0, t) - \phi_{20}(0, t)]$$

$$+ h_0[T_1(0, t) - T_2(0, t)] \tag{A-6}$$

$$\phi_{10}(x, 0) = 0 \tag{A-7}$$

$$-k_1 \frac{\partial \phi_{10}}{\partial x}\bigg|_{x=-L} = A_1(t)\, \phi_{10}(-L, t). \tag{A-8}$$

Now replace $h(t) = h_0$ by a series of functions $h^i, i = 1, 2, \ldots, M$ where

$$h(t) = h^i = h_0 \qquad \text{for} \quad t_i < t < t_{i+1}. \tag{A-9}$$

Define

$$\phi_1^i \equiv h^i \frac{\partial T_1}{\partial h^i}; \qquad \phi_2^i \equiv h^i \frac{\partial T_2}{\partial h^i} \tag{A-10}$$

and take the partial derivative of equations (1), (3), (A-1) and (A-3) with respect to h^i and then multiply by $h^i = h_0$ to obtain

$$k_1 \frac{\partial^2 \phi_1^i}{\partial x^2} = \rho_1 c_{p,1} \frac{\partial \phi_1^i}{\partial t} \tag{A-11}$$

$$-k_1 \frac{\partial \phi_1^i}{\partial x}\bigg|_{x=0} = h_0[\phi_1^i(0, t) - \phi_2^i(0, t)]$$

$$+ h_0 \delta(t)\, [T_1(0, t) - T_2(0, t)] \tag{A-12}$$

$$\phi_1^i(x, 0) = 0 \tag{A-13}$$

$$-k_1 \frac{\partial \phi_1^i}{\partial x}\bigg|_{x=-L} = A(t)\, \phi_1^i(-L, t) \tag{A-14}$$

where
$$\delta(t) = 1 \text{ if } t_i < t < t_{i+1} \quad = 0 \text{ otherwise.} \tag{A-15}$$

If now equation (A-11) is written for $i = 1, 2, \ldots, M$ and the equations are added together, there results

$$k_1 \frac{\partial^2 G}{\partial x^2} = \rho_1 c_{p,1} \frac{\partial G}{\partial t} \tag{A-16}$$

where

$$G = \sum_{i=1}^{M} \phi_1^i. \tag{A-17}$$

Repeat the summation procedure for equations (A-12)–(A-14). Comparing these equations with (A-5)–(A-8) shows that

$$\phi_{10}(x, t) = G = \sum_{i=1}^{M} \phi_1^i. \tag{A-18}$$

A similar result can be demonstrated for body 2. These latter two results yield equation (11). Note that equation (11) is valid for a variety of initial and boundary conditions; the basic restrictions are that the thermal properties are temperature-independent (i.e. the problem must be linear).

Résumé—Certains travaux récents indiquent que la conductance du contact thermique peut varier d'une façon importante avec le temps. Dans les études antérieures des cas transitoires, les analyses étaient restreintes aux cas utilisant une spécimen mince ou des thermocouples placés à l'interface. Comme la conductance du contact est un effet de volume, ces restrictions sont fréquemment trop sévères. Pour cette raison, on présente une méthode pour l'analyse des résultats de température obtenus à partir des spécimens thermiquement épais avec les thermocouples placés à l'interface.

Il y a un certain nombre d'expériences transitoires possibles qui pourraient être employées, mais elles ne sont pas toutes également aussi efficaces pour déterminer la conductance. On donne un critère pour permettre les comparaisons entre les expériences; un certain nombre d'expériences possibles est comparé en employant ce critère. A partir de cette comparaison quelques expériences optimales sont indiquées. Ces expériences optimales permettent de déterminer la conductance de contact plus précisément qu'en employant d'autres expériences semblables avec la même précision des mesures de température.

Zusammenfassung—Kürzlich erschienene Arbeiten deuten an, dass sich der thermische Übergangswiderstand stark mit der Zeit ändern kann. In früheren Untersuchungen von instationären Fällen waren die Analysen auf dünne Proben beschränkt oder auf Fälle in welchen die Thermoelemente in der Trennfuge angeordnet waren. Da der Übergangswiderstand ein Volumeneffekt ist, sind diese Einschränkungen häufig zu stark. Daher wird eine Methode angegeben, zur Analysierung der Temperaturen die an dicken Proben mit Thermoelementen die nicht in der Trennfuge lagen, gemessen wurden.

Es können eine Reihe von möglichen instationären Versuchen angewandt werden, jedoch sind sie zur Bestimmung des Widerstandes nicht gleich wirksam. Zum Vergleich der einzelnen Versuche wird ein Kriterium angegeben und eine Anzahl von Versuchen wird damit beurteilt. Es ergeben sich einige optimale Versuche. Sie erlauben eine genauere Bestimmung des Übergangswiderstandes als andere, ähnliche Versuche, deren Temperaturmessungen gleich genau durchgeführt wurden.

Аннотация—В некоторых последних работах указывается, что контактная теплопроводимость может значительно изменяться со временем. В предыдущих исследованиях нестационарных процессов анализы были ограничены случаями, в которых использовался или тонкий образец или термопары, расположенные на границе раздела. Так как контактная проводимость представляет собой объемный эффект, эти ограничения являются слишком строгими. По этой причине представлен метод для анализа температурных данных, полученных на термически толстых образцах, когда термопары не обязательно распологались на границе раздела.

Можно провести ряд экспериментов по нестационарной теплопроводности, которые могли бы быть использованы, но они не всегда будут одинаково эффективными при определении проводимости. Представлен критерий, позволяющий сравнить эксперименты ; с помощью этого критерия проведено сравнение нескольких возможных экспериментов ; приводится сравнение ряда экспериментов на основе использования этого критерия. В результате отобраны некоторые оптимальные эксперименты, позволяющие определить контактную проводимость более тщательно, чем с помощью других аналогичных экспериментов, в которых измерения температуры производились с такой же точностью.